Int. J. Heat Mass Transfer. Vol. 35, No. 12, pp. 3389–3395, 1992
Printed in Great Britain

0017-9310/92 $5.00+0.00
Pergamon Press Ltd

Concerning the calculation of plane turbulent jets on the basis of the $k-\varepsilon$ model of turbulence

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(*Received* 16 *July* 1991)

Abstract—Within the framework of the standard $k-\varepsilon-\langle T'^2\rangle$ model of turbulence, self-similar equations for the velocity and temperature fields of a plane jet have been integrated numerically. Tables of solutions for velocity, temperature, kinetic energy of turbulence, rate of its dissipation, and for the mean square of temperature fluctuation at different Prandtl numbers are presented. The quantitative parameters for the evolution of mean and turbulent characteristics of free jet flows have been determined. A new numerical solution scheme for a non-similar partial differential equation is described. Comparison between the results obtained and experimental/numerical data of other authors is made.

1. INTRODUCTION

JET LIQUID and gas flows play an important role in natural and engineering hydrodynamic systems. Because of this, for more than a century the theory of turbulent jets and its practical applications have attracted the attention of specialists working not only in the field of hydromechanics, but also in such branches of science as oceanology, meteorology, physics of the atmosphere, etc. The interest to the problem of turbulent jet flows was spurred at the end of the 1970s–beginning of the 1980s by the rapid development all over the world of power engineering and industrial production accompanied by the growth of the large amounts of technological waters and various kinds of combustion products that usually enter the ambient water and air media in the form of jet wastes. The understanding and prediction of the laws that govern momentum and heat transfer in discharged streams allows one not only to predict the zones with elevated concentrations of particular substances and their lifetime under different conditions, but also to control the process itself. The latter stimulates the development and refinement of the mathematical models and methods for analyzing different types of jet flows.

For a long time, the study of laws governing the development of turbulent jets, along with the experimental ways of solving the problem, was based on the so-called gradient models of turbulence that relied on the analogy with laminar flows. These models made it possible to quite generally describe a particular phenomenon, but they involved quite a number of arbitrary 'floating' constants which were determined in the process of matching the predicted results with experimental data. As a result, the applicability and use of the solutions constructed was usually limited by those conditions under which the test data were obtained.

This has compelled many scientists to undertake the development of new, more improved models which employ differential equations of the kinetic energy k and of the rate of its dissipation ε in the closure problem. Since in the majority of applied studies of turbulent shear flows the $(k-\varepsilon)$-model made it possible to combine a relative simplicity with a satisfactory physical realism, it has become most popular at the present time. Checks have revealed that it is more general than gradient models, since with the same initial empirical information it allows one to calculate a number of different engineering problems (jets, wakes, channel flows, etc.).

However, as regards the quantitative results given in the available publications (a certain stage in the development of the theory of turbulent jets has received its treatment in ref. [1]), it turns out very often that the numerical data obtained in the analysis of identical problems do not agree amongst themselves. Naturally, this results from the employment of different approaches to numerical solution (selection of a numerical algorithm), the number and distribution of grid points, limitation on the computational domain of the flow from different approximations of differential equations proper and also from the somewhat different initial empirical coefficients.

From this it follows, in particular, that a good coincidence of any computational method with the results of one experiment or with a small group of tests does not always guarantee its reliability.

Moreover, the unavailability of a standard introduces arbitrariness not only in the interpretation of quantitative results obtained in different works, but of the possibilities of the mathematical model as a whole. In these conditions, of great theoretical and applied importance are the approaches to the analysis of the problems which allow one to readily distinguish the errors introduced by numerical procedures from those associated with the simulation of turbulence. In

NOMENCLATURE

b	jet half-width
C_p	specific heat at constant pressure
K	flow momentum
k	kinetic energy of turbulence
Q	heat flux
q	mean square of temperature fluctuation, $\langle T'^2 \rangle$
r_0	jet width at exit
T	temperature, $\Delta T = T - T_\infty$
T'	temperature fluctuation
u'	turbulent fluctuations of velocity u
v'	turbulent fluctuations of velocity v
x	longitudinal coordinate
y	transverse coordinate.

Greek symbols

ε	dissipation rate
ζ	self-similar coordinate
ν_t	turbulent viscosity coefficient
ρ	density
σ_t	turbulent Prandtl number.

Subscripts

∞	surrounding fluid
c	axial line of jet
$*$	jet boundary
0	exit jet cross-section
u	refers to velocity
θ	refers to temperature.

particular, one such approach is the deviation and construction of self-similar solutions for different kinds of models [2, 3]. From a mathematical viewpoint, the self-similar solutions are convenient because partial differential equations can be reduced to ordinary differential equations, the solution of which can be obtained with a high accuracy.

In the present paper, self-similar equations for the scalar field of a plane free jet, which are based on the use of the $k-\varepsilon-\langle T'^2\rangle$ model of turbulence, are given and numerically integrated. Detailed tables of the results obtained are presented which can be used for evaluating the accuracy of numerical data obtained within the framework of ordinary finite-difference schemes for the asymptotic area of a plane jet flow. A new effective method for solving a non-self-similar problem is suggested which is based on the introduction of mathematical variables in which the problem of finding unknown functions, that describe the development of flow from the profiles prescribed at the nozzle tip to the asymptotic distributions, is reduced to the analysis of a system of first-order ordinary differential equations. Comparison of the results of the present calculations with self-similar solutions and also with numerical and experimental data of other authors is carried out.

2. STATEMENT OF THE PROBLEM

Let u and v be the averaged velocity components directed along (x) and normally to (y) the jet axis. It is assumed that the flow is steady and that it satisfies the boundary layer approximation. The Reynolds numbers are regarded to be rather high, so that the effect of molecular viscosity in explicit form has turned out to be insignificant. Then, the equations of continuity, momentum and energy can be written down in the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\nu_t\frac{\partial u}{\partial y}\right)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{1}{\sigma_t}\frac{\partial}{\partial y}\left(\nu_t\frac{\partial T}{\partial y}\right). \qquad (1)$$

With the use of the $k-\varepsilon-\langle T'^2\rangle$-model of turbulence, the main quantities that characterize the turbulent transfer are taken to be the local values of the kinetic energy k, rate of its dissipation ε and the mean square of temperature fluctuations $\langle T'^2\rangle$, which satisfy the following model equations:

$$u\frac{\partial k}{\partial x}+v\frac{\partial k}{\partial y}=\frac{1}{\sigma_k}\frac{\partial}{\partial y}\left(\nu_t\frac{\partial k}{\partial y}\right)+\nu_t\left(\frac{\partial u}{\partial y}\right)^2-\varepsilon$$

$$u\frac{\partial \varepsilon}{\partial x}+v\frac{\partial \varepsilon}{\partial y}=\frac{1}{\sigma_\varepsilon}\frac{\partial}{\partial y}\left(\nu_t\frac{\partial \varepsilon}{\partial y}\right)+c_{\varepsilon 1}\frac{\varepsilon}{k}\nu_t\left(\frac{\partial u}{\partial y}\right)^2-c_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

$$u\frac{\partial\langle T'^2\rangle}{\partial x}+v\frac{\partial\langle T'^2\rangle}{\partial y}=\frac{1}{\sigma_q}\frac{\partial}{\partial y}\left(\nu_t\frac{\partial\langle T'^2\rangle}{\partial y}\right)$$
$$+2\frac{\nu_t}{\sigma_t}\left(\frac{\partial T}{\partial y}\right)^2-c_{q1}\frac{\varepsilon}{k}\langle T'^2\rangle \qquad (2)$$

where $c_{\varepsilon 1}$, $c_{\varepsilon 2}$, and c_{q1} are additional constants of the problem. Moreover, σ_k, σ_ε, σ_t, and σ_q are the turbulent Prandtl numbers for k, εT, and $\langle T'^2\rangle$, respectively. Next, the model is closed by an expression for turbulent viscosity

$$\nu_t=c_\mu\frac{k^2}{\varepsilon}. \qquad (3)$$

Equations (1)–(3) form a system of partial differential equations which involves seven empirical constants

$$c_\mu=0.09,\quad c_{\varepsilon 1}=1.44,\quad c_{\varepsilon 2}=1.92,\quad \sigma_k=1.0,$$
$$\sigma_\varepsilon=1.3,\quad c_{q1}=1.25,\quad \sigma_q=0.6923. \qquad (4)$$

And, finally, the statement of the problem is completed with the formulation of initial and boundary conditions for the velocity and temperature fields of a plane free jet

$$x=0\begin{cases} u=u_0,\quad T=T_0,\ k=k_0,\quad \varepsilon=\varepsilon_0,\quad q=q_0 \\ \quad\text{when}\quad 0\leqslant y<\dfrac{r_0}{2}, \\ u=0,\quad T=T_\infty,\quad k=0,\quad \varepsilon=0,\quad q=0 \\ \quad\text{when}\quad \dfrac{r_0}{2}\leqslant y<\infty \end{cases} \tag{5a}$$

$$x>0\begin{cases} y=0:\quad v=\dfrac{\partial u}{\partial y}=\dfrac{\partial k}{\partial y}=\dfrac{\partial \varepsilon}{\partial y}=\dfrac{\partial T}{\partial y}=\dfrac{\partial q}{\partial y}=0 \\ y\to y_*:\quad u,k,\varepsilon,q\to 0,\ T\to T_\infty. \end{cases} \tag{5b}$$

3. SELF-SIMILAR SOLUTIONS

Since this section will deal with the character of change of the basic quantities for the asymptotic (self-similar) region of the jet flow development, initial conditions (5a) should be replaced by the integral relations

$$K_0=2\rho\int_0^{y_*}u^2\,\mathrm{d}y,\quad Q_0=2\rho C_p\int_0^{y_*}u\Delta T\,\mathrm{d}y \tag{6}$$

that express the conservation laws for momentum and energy fluxes through a plane perpendicular to the jet axis. The solution of the above-stated problem will be sought in the form

$$\psi=\left(\frac{K_0\sqrt{c_\mu}}{\rho}\right)^{1/2}f(\zeta)x^{1/2},\quad k=\frac{K_0}{\rho\sqrt{c_\mu}}a(\zeta)x^{-1}$$

$$\varepsilon=\left(\frac{K_0}{\rho\sqrt{c_\mu}}\right)^{3/2}b(\zeta)x^{-5/2},\quad \zeta=\frac{y}{\sqrt{c_\mu}x}$$

$$\Delta T=\frac{Q_0}{\rho C_p}\left(\frac{\rho}{K_0\sqrt{c_\mu}}\right)^{1/2}h(\zeta)x^{-1/2}$$

$$\langle T'^2\rangle=\frac{Q_0^2}{\rho C_p^2K_0\sqrt{c_\mu}}c(\zeta)x^{-1} \tag{7}$$

where $f(\zeta)$, $a(\zeta)$, $b(\zeta)$, $c(\zeta)$, and $h(\zeta)$ are unknown functions of the self-similar variable ζ. By virtue of the continuity equation and the formulae of transition from the former coordinates x and y to the new coordinates x and ζ, one obtains

$$u=\left(\frac{K_0}{\rho\sqrt{c_\mu}}\right)^{1/2}f'x^{-1/2}. \tag{8}$$

Note that in similarity variables, equations (7), (8), the shapes of the profiles are independent of the constant c_μ. Now, substituting equations (7), (8) into equations (1)–(3), (5b), and (6), a system of ordinary differential equations will be written by the standard technique:

$$\left(\frac{a^2}{b}f''\right)'+\tfrac{1}{2}ff''+\tfrac{1}{2}f'^2=0$$

$$\frac{1}{\sigma_k}\left(\frac{a^2}{b}a'\right)'+\tfrac{1}{2}fa'+f'a+\frac{a^2}{b}f''^2-b=0$$

$$\frac{1}{\sigma_t}\left(\frac{a^2}{b}h'\right)'+\tfrac{1}{2}fh'+\tfrac{1}{2}f'h=0$$

$$\frac{1}{\sigma_\varepsilon}\left(\frac{a^2}{b}b'\right)'+\tfrac{1}{2}fb'+\tfrac{5}{2}f'b+c_{\varepsilon1}af''^2-c_{\varepsilon2}\frac{b^2}{a}=0$$

$$\frac{1}{\sigma_q}\left(\frac{a^2}{b}c'\right)'+\tfrac{1}{2}fc'+\left(f'-c_{q1}\frac{b}{a}\right)c+\frac{2}{\sigma_t}\frac{a^2}{b}h'^2=0 \tag{9a}$$

with boundary conditions

$$\zeta=0:\quad f=f''=a'=b'=c'=h'=0$$

$$\zeta\to\zeta_*:\quad f',a,b,c,h\to 0 \tag{9b}$$

and integral conditions

$$2\int_0^{\zeta_*}f'^2\,\mathrm{d}\zeta=1,\quad 2\int_0^{\zeta_*}f'h\,\mathrm{d}\zeta=1. \tag{9c}$$

Relations (9) form an essentially non-linear two-point boundary-value problem in which the value of

$$\zeta_*=\frac{y_*}{\sqrt{c_\mu}x}$$

(y_* is the line separating the turbulent zone from the region of irrotational flow) is also unknown and should be found in the process of integration. Moreover, the analysis of the problem is complicated by the existence of a singularity at the jet boundary ζ_* due to the tending to zero of the kinetic energy k and of the rate of its dissipation ε (it has been ascertained in ref. [3] that $v_t\to 0$ when $\zeta\to\zeta_*$).

The system of equations at different values of the turbulent Prandtl number σ_t was solved numerically on the basis of the Runge–Kutta scheme with an automatic selection of the integration step. Calculation started at $z=0$ and continued up to $z=z_\infty$ ($z_\infty=72$), which is a numerical approximation of the mathematical point $z=\infty$.

Here z is a new variable

$$z=\int_0^\zeta\frac{b}{a^2}\,\mathrm{d}\zeta \tag{10}$$

which not only eliminates the singularities of equations (9), but also simplifies the numerical integration procedure: since the new coordinate involves the dependence of the turbulent viscosity in ζ, equations (9) in the variable z already look like ordinary bound-

Table 1. Results of numerical solution of self-similar equations at different Prandtl numbers

σ_t	$f''(0)$	$a(0)$	$b(0)$	$\zeta_{0.5u}$	$\left[-\frac{a^2}{b}f''\right]_m$	$h(0)$	$c(0)$	c_m	$\zeta_{0.5\theta}$	$\left[-\frac{a^2h'}{\sigma_t b}\right]_m$
0.5						1.17926	0.113819	0.1833	0.5243	0.1642
0.55						1.19938	0.120264	0.1842	0.4992	0.1614
0.6						1.21927	0.126585	0.1862	0.4768	0.1589
	1.37145	0.123527	0.300437	0.3600	0.1459					
0.65						1.23896	0.132803	0.1886	0.4568	0.1567
0.7						1.25844	0.138927	0.1920	0.4388	0.1547
0.75						1.27772	0.144969	0.1954	0.4225	0.1529

ary layer equations involving a constant turbulent viscosity with $z \to \infty$ when $\zeta \to \zeta_*$. The lacking boundary conditions, i.e. $f''(0)$, $a(0)$, $b(0)$, $c(0)$, $h(0)$, were found with the aid of the shooting technique; iterations were continued to the convergence 1×10^{-4} over all the unknown functions at the conventional boundary of the jet z_{γ}. Basic results of the calculations are presented in Table 1. Now, prescribing the value of c_μ equal to 0.09 yields the following expressions

$$u_c = A_u\left(\frac{K_0}{\rho}\right)^{1/2} x^{-1/2},\quad \Delta T_c = \frac{Q_0}{\rho C_p}\left(\frac{\rho}{K_0}\right)^{1/2} A_\theta x^{-1/2}$$

$$k_c = A_k\left(\frac{K_0}{\rho}\right)x^{-1},\quad \varepsilon_c = A_\varepsilon\left(\frac{K_0}{\rho}\right)^{3/2} x^{-5/2} \qquad (11)$$

where the values of the constants A_u, A_k, A_ε, and A_θ are listed in Table 2. Formulae (11) represent a certain standard with the aid of which one may judge the accuracy of solutions obtained for the asymptotic region of a plane jet on the basis of the $k-\varepsilon$-model of turbulent numerical schemes, as well as distinguish the errors introduced by numerical procedures from those introduced by the models.

As to the lateral distribution of turbulent transfer fluxes

$$\frac{\langle u'v'\rangle}{u_c^2} = -\sqrt{c_\mu}\,\frac{\frac{a^2}{b}f''}{[f''(0)]^2},$$

$$\frac{\langle v'T'\rangle}{u_c\Delta T_c} = -\frac{\sqrt{c_\mu}}{\sigma_t}\,\frac{\frac{a^2}{b}h'}{f''(0)h(0)}$$

these have the form of conventional free jet flows. For example, the maximum value of $\langle u'v'\rangle/u_c^2$ at $c_\mu = 0.09$ is 0.0233 which is attained at $\zeta/\zeta_{0.5u} \approx 0.85$. The peak value of the turbulent heat flux, normalized, as is usually the case, by $u_c\Delta T_c$, is somewhat higher than $\langle u'v'\rangle_m/u_c^2$ and amounts to 0.0285 at $\zeta/\zeta_{0.5u} \approx 0.94$ ($\sigma_t = 0.6$). A clear dependence of $\langle v'T'\rangle_m/u_c\Delta T_c$ on σ_t is being observed: with increase of the latter value, the former value decreases, whereas the maximum point shifts to the flow axis.

From the viewpoint of the analysis and construction of a more complex model of turbulence, a very important characteristic is $\langle T'^2\rangle$. The computational information obtained about the self-similar profile of temperature fluctuations indicates (see Table 2) that the values of $\sqrt{(q_c)}/\Delta T_c$ and $\sqrt{(q_m)}/\Delta T_c$ turn out to be higher than those experimentally measured, i.e. $\sqrt{(q_c)}/\Delta T_c = 0.22$ and $\sqrt{(q_m)}/\Delta T_c = 0.30$ [4]. Therefore, an attempt has also been made in the present work to find a set of constants—σ_q, c_{q1}—that determine the self-similar solutions which would most closely correspond to experimental data. Numerical integration was made at those values of σ_q and c_{q1} which are already employed in calculations of jet flows [5]. It has turned out that the best prediction is provided by the following set of coefficients: $\sigma_q = 0.6923$ and $c_{q1} = 1.79$ (Table 3). In this case the relative magnitude of the intensity of temperature fluctuations comprised 0.213 on the axis and 0.296 at $\zeta/\zeta_{0.5u} \approx 1.13$ ($\sigma_t = 0.6$).

A sharp decrease of $\sqrt{q}$ near the jet axis is usually typical of those types of jet flows in which the buoyancy effects are absent and the equilibration of tem-

Table 2. Self-similar characteristics of a plane jet at different turbulent Prandtl numbers

σ_t	A_u	A_k	A_ε	b_u/x	k_c/u_c^2	$\frac{\langle u'v'\rangle_m}{u_c^2}$	A_θ	$\frac{\sqrt{\langle T'^2\rangle_c}}{\Delta T_c}$	$\frac{\sqrt{\langle T'^2\rangle_m}}{\Delta T_c}$	b_θ/x	$\frac{\langle v'T'\rangle_m}{u_c\Delta T_c}$
0.5							2.15	0.29	0.36	0.157	0.030
0.55							2.19	0.29	0.36	0.150	0.029
0.6							2.23	0.29	0.35	0.143	0.0285
	2.50	0.41	1.83	0.108	0.0657	0.023					
0.65							2.26	0.29	0.35	0.137	0.028
0.7							2.30	0.30	0.35	0.132	0.027
0.75							2.33	0.30	0.35	0.127	0.026

Table 3. Results of numerical solution of self-similar equations for the temperature field of a plane turbulent jet

σ_t	0.5					0.6				
σ_q	0.6923			1.0		0.6923			1.0	
c_{q1}	1.25	1.54	1.79	1.25	1.54	1.25	1.54	1.79	1.25	1.54
$\dfrac{\sqrt{\langle T'^2\rangle_c}}{\Delta T_c}$	0.29	0.24	0.21	0.27	0.22	0.29	0.24	0.21	0.28	0.23
$\dfrac{\sqrt{\langle T'^2\rangle_m}}{\Delta T_c}$	0.36	0.33	0.31	0.37	0.33	0.35	0.32	0.30	0.36	0.32

peratures is associated in the main only with hydrodynamic agitation in the mixing layer.

To determine the shear layer boundary ζ_*, use will be made of the formula

$$\zeta_* = \int_0^\infty \frac{a^2}{b}\,\mathrm{d}z$$

by virtue of which $\zeta_* = 0.8292$ (in ref. [3] the value obtained was $\zeta_* = 0.83127$). Thus, according to the $k-\varepsilon$ model of turbulence, there is a line which separates the turbulent zone from the irrotational flow region. The latter result testifies to the fact that it is necessary to more adequately approach the problem of specifying the artificial boundaries which are to be introduced into numerical analysis for limiting the computational domains of free jet flows.

4. NON-SELF-SIMILAR SOLUTIONS

The drawback of equations (11) is that the region of their applicability is limited by an asymptotic portion in the development of a plane forced jet, which is observed, according to ref. [6], when $x/r_0 > 10$ for mean parameters and when $x/r_0 > 20$ for pulsational parameters. Therefore, it is natural that one cannot except that equations (11) can ensure an exact quantitative description of the basic features of turbulent flow for $x/r_0 < 10$.

These limitations can be overcome only by constructing non-self-similar solutions that take into account the dependence of the unknown functions on the initial parameters of the problem. This implies a direct numerical integration of the system of partial differential equations (1)–(3) with provision for equations (4), (5), for example, by the finite-difference method. In this case, the flow region ($0 < x < \infty$, $0 \leqslant y < \infty$) is overlaid with a rectangular grid with the step Δx, Δy at the nodes of which the differential equations are replaced by difference analogs. As a result, the initial expressions are reduced to an approximating system of algebraic equalities which are solved by the trial-run method with the use of iterations.

In the present paper an alternative approach to the analysis of problems (1)–(5), is suggested. Having normalized the unknown functions

$$U = \frac{u}{u_0},\quad V = \frac{v}{u_0},\quad K = \frac{k}{u_0^2},\quad E = \frac{\varepsilon r_0}{u_0^3}$$

$$X = \frac{x}{r_0},\quad Y = \frac{y}{r_0},\quad \theta = \frac{T-T_\infty}{T_0-T_\infty}$$

we turn to new variables

$$X = X,\quad \eta = 2\int_0^Y U^2\,\mathrm{d}Y \tag{12a}$$

on the basis of the formulae

$$\frac{\partial}{\partial Y} = 2U^2\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial X} + \left(4c_\mu\frac{K^2U^2}{E}\frac{\partial U}{\partial \eta} - 2UV\right)\frac{\partial}{\partial \eta}. \tag{12b}$$

By virtue of equations (12) one obtains

$$\frac{\partial U}{\partial X} = 4c_\mu\frac{K^2U^2}{E}\left[U\frac{\partial^2 U}{\partial \eta^2} + \left(\frac{\partial U}{\partial \eta}\right)^2 + 2\frac{U}{K}\frac{\partial K}{\partial \eta}\frac{\partial U}{\partial \eta} - \frac{U}{E}\frac{\partial E}{\partial \eta}\frac{\partial U}{\partial \eta}\right]$$

$$\frac{\partial K}{\partial X} = 4\frac{c_\mu}{\sigma_k}\frac{U^2K^2}{E}\left[U\frac{\partial^2 K}{\partial \eta^2} + 2\frac{U}{K}\left(\frac{\partial K}{\partial \eta}\right)^2 + \sigma_k U\left(\frac{\partial U}{\partial \eta}\right)^2 + (2-\sigma_k)\frac{\partial U}{\partial \eta}\frac{\partial K}{\partial \eta} - \frac{U}{E}\frac{\partial K}{\partial \eta}\frac{\partial E}{\partial \eta} - \frac{\sigma_k}{4c_\mu}\frac{E^2}{K^2U^3}\right]$$

$$\frac{\partial E}{\partial X} = 4\frac{c_\mu}{\sigma_\varepsilon}\frac{U^2K^2}{E}\left[U\frac{\partial^2 E}{\partial \eta^2} + \sigma_\varepsilon c_{\varepsilon 1}\frac{UE}{K}\left(\frac{\partial U}{\partial \eta}\right)^2 + (2-\sigma_\varepsilon)\frac{\partial U}{\partial \eta}\frac{\partial E}{\partial \eta} + 2\frac{U}{K}\frac{\partial K}{\partial \eta}\frac{\partial E}{\partial \eta} - \frac{U}{E}\left(\frac{\partial E}{\partial \eta}\right)^2 - \frac{\sigma_\varepsilon c_{\varepsilon 2}}{4c_\mu}\frac{E^3}{U^3K^3}\right]$$

$$\frac{\partial \theta}{\partial X} = 4\frac{c_\mu}{\sigma_t}\frac{U^2K^2}{E}\left[U\frac{\partial^2 \theta}{\partial \eta^2} + 2\frac{U}{K}\frac{\partial K}{\partial \eta}\frac{\partial \theta}{\partial \eta} + (2-\sigma_t)\frac{\partial U}{\partial \eta}\frac{\partial \theta}{\partial \eta} - \frac{U}{E}\frac{\partial E}{\partial \eta}\frac{\partial \theta}{\partial \eta}\right]. \tag{13}$$

Since during the change of the coordinate Y from 0

Table 4. Comparison of predicted and experimental asymptotic characteristics of a plane turbulent jet

Reference	A_u	A_k	A_ε	k_c/u_c^2	b_u/x	$\frac{\langle u'v'\rangle_m}{u_c^2}$	A_θ	b_θ/x	$\frac{\langle v'T'\rangle_m}{u_c\Delta T_c}$
[5]	2.37	0.38	1.55	0.068	0.116		2.10	0.154	
[9]	2.35	0.37	1.69	0.067	0.100		2.02	0.133	
[10]	2.40	0.35		0.061		0.025			
Present work	2.57	0.39	1.78	0.059	0.103	0.021	2.15	0.137	0.028
[7]				0.067	0.109	0.026			
[6]					0.112	0.020	2.27	0.167	0.018
[8]	2.48				0.096		2.00	0.142	

to ∞ the value of η varied from 0 to $k_0/\rho u_0^2 r_0$, equalities (13) determine the unknown functions in the region $X \geqslant 0$, $0 \leqslant \eta < K_0/\rho u_0^2 r_0 = I$ (when assigning the uniform velocity field at the slit cut one obtains that $X \geqslant 0$, $0 \leqslant \eta < 2$). Consequently, instead of integrating the set problems (1)–(5) in an infinite region, we obtain equations (13) in a rectangle. Then, through the points of the straight line η draw a system of n lines parallel to the axis X at I/n intervals. On each of the straight lines $\eta = \eta_i$ $(i = 1, 2, \ldots, n+1)$, the derivatives with respect to η will be replaced by their three-point central-difference ratios. As a result, when prescribing the initial conditions at $X = 0$ or in some section X^* (these can be model conditions or those taken directly from the experiment), equations (13) transform into a system of first-order ordinary differential equations (Cauchy problem), which can be integrated with the aid of a standard Runge–Kutta method with the step along the coordinate X selected automatically.

Let us now direct our attention to the analysis of the results of numerical calculus. Calculations were begun in the exit section of the slit where the uniform profiles of velocity, temperature, of the kinetic energy of turbulence and of the rate of its dissipation were assumed

$$U_0 = 1.0, \quad \theta_0 = 1.0, \quad K_0 = 0.02, \quad E_0 = 0.0016.$$

Then the development of a plane jet up to the section $X = 100$ was calculated. It has been found that starting from $X \approx 20$ the behaviour of the sought-after solutions on the flow axis can be approximated by the power functions of type (11), where the values of the found coefficients A_u, A_θ, A_k, and A_ε are presented in Table 4 ($\sigma_t = 0.6$). Note that the transition from the mathematical variables X, η to the physical variables X, Y is performed by the equation

$$Y = \frac{1}{2}\int_0^\eta \frac{d\eta}{U^2}.$$

Analysis of the comparison of the results of calculation with self-similar solutions (Table 2) shows that the proposed method of numerical analysis rather accurately predicts the asymptotic behaviour of the main parameters of the $(k-\varepsilon)$-model (deviations do not exceed 5%), even in the case when the flow field is roughly divided into bands (it was assumed in calculations that $n = 50$). The quantitative discrepancy decreases with the growth of the number n. In our opinion, this results from the fact, that in the numerical scheme the provision is made for the automatic fulfilment of the jet momentum conservation law K_0 over the enitre computational domain, therefore even with a rough step $\Delta\eta$ the solution turns out to be qualitatively correct and rather accurate.

Also compared were the results of numerical integration of the non-self-similar problem with the experimental data of refs. [6–8], and their satisfactory agreement over a broad spectrum of characteristics was noted (see Table 4). An appreciable discrepancy is observed only in the evaluation of the heat flux distribution $\langle v'T'\rangle_m/u_c\Delta T_c$ in the transverse direction: the maximum value for the asymptotic region of flow development is equal to 0.0285, whereas that measured amounts only to 0.018 [6] and, moreover, it is smaller than the experimental value obtained for $\langle u'v'\rangle_m/u_c^2 = 0.020$. Such relationship between the maximum values of turbulent fluxes normalized, as usual, by u_c^2 and $u_c\Delta T_c$, is obtained in the $(k-\varepsilon)$-model only when $\sigma_t > 1$, which does not account for the physics of the phenomenon: as is known, in turbulent forced jets $\sigma_t < 1$.

And, finally, Table 4 contains numerical estimates of the plane forced jet structure obtained on the basis of the $(k-\varepsilon)$-model of turbulence within the frameworks of different numerical schemes [5, 9, 10].

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